

Geometric flow of G_2 -structures on $C(S^3 \times S^3)^*$

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Abstract

We introduce a first order flow of G_2 -structures and construct its explicit solution in case of a cone over $S^3 \times S^3$. Also we prove for this situation that starting from certain initial datum the flow deforms corresponding to G_2 -structure metric to a conic metric up to homotheties.

1 Introduction

Theory of flows of G_2 -structures has appeared quite recently. The most famous examples of these flows are Laplacian flow that was introduced by R. Bryant in [3] and S. Karigiannis' General flow [4]. It's not clear whether there exists a 'distinguished' flow of G_2 -structures that would lead to a parallel structure on manifolds satisfying some (still unknown) conditions.

In this article we construct a first order flow of G_2 -structures that possesses interesting solutions in special cases. We find explicit solution of this flow in case when our manifold is a cone over $S^3 \times S^3$ given by certain family of G_2 -structures.

For mentioned manifold we show that corresponding to G_2 -structure metric satisfying some conditions evolves along the flow to a conic metric up to rescalings at every moment of time.

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2 Definition of the geometric flow

Let's consider 8-dimensional octonion algebra \mathbb{O} with a basis $1, e_1, e_2, \dots, e_7$ and a multiplication law shown on fig. 1. We will identify imaginary $\text{Im}(\mathbb{O}) = \text{Span}(e_1, e_2, \dots, e_7)$ octonions and \mathbb{R}^7 . Multiplication \circ in \mathbb{O} defines positive definite $\langle \cdot, \cdot \rangle$ and cross $\cdot \times \cdot$ products on pairs of vectors $u, v \in \mathbb{R}^7 \cong \text{Im}(\mathbb{O})$ as follows:

$$\langle u, v \rangle = -\text{Re}(u \circ v)$$

$$u \times v = \text{Im}(u \circ v)$$

These products allow us to define a 3-form (associative form) ϕ on \mathbb{R}^7 by the formula

$$\phi(u, v, w) = \langle u \times v, w \rangle$$

This form is non-degenerate in the next sense: for any $x, y \in \mathbb{R}^7$

$$(x \lrcorner \phi) \wedge (y \lrcorner \phi) \wedge \phi \neq 0$$

Written in the basis e_1, e_2, \dots, e_7 it looks as follows:

$$\phi = e^{456} + e^{621} + e^{174} + e^{527} + e^{637} + e^{135} + e^{432}$$

where by e^{ijk} we denote the basic form $e^i \wedge e^j \wedge e^k$ and $e^i(e_j) = \delta_j^i$.

Definition 1 *The subgroup of GL_7 that fixes (with respect to canonical action $GL_7 \hookrightarrow \wedge^3(\mathbb{R}^7)$) the non-degenerate form ϕ is called the group G_2 .*

It is 14-dimensional simple Lie group of type G_2 . The orbit $\wedge_+^3(\mathbb{R}^7)$ of ϕ that consists of non-degenerate 3-forms is open in $\wedge^3(\mathbb{R}^7)$ because

$$\dim \wedge_+^3(\mathbb{R}^7) = \dim GL_7 - \dim G_2 = \dim \wedge^3(\mathbb{R}^7)$$

Henceforward M is a 7-dimensional manifold. Differential 3-form φ on M is called non-degenerate if $\varphi(x)$ is non-degenerate as a 3-form on $T_x(M) \forall x \in M$.

Definition 2 *Any non-degenerate 3-form φ on M will be called a G_2 -structure on M .*

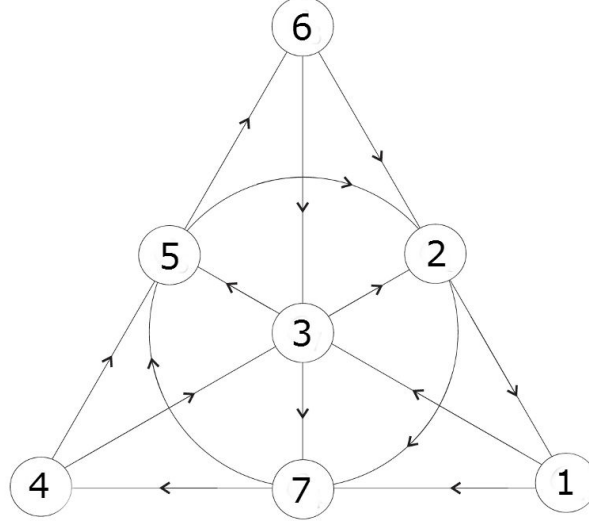


Figure 1: Fano plane

In a neighborhood of a point $p \in M$ such local coordinates could be chosen that in these coordinates $\varphi(p)$ coincides with above described associative form ϕ . The set $\Lambda_+^3(M)$ of non-degenerate 3-forms on M is just the set of smooth sections of the bundle $\Lambda_+^3(TM)$ over M with a fiber $\Lambda_+^3(\mathbb{R}^7)$. It is known that M admits a G_2 -structure if and only if the first two Stiefel-Whitney classes of M vanish.

G_2 -structure on M allows one to define riemannian metric $g = g_\varphi$ on M as it follows. In local coordinates x^1, x^2, \dots, x^7 in a neighborhood of point $p \in M$ let's define the tensor field B by the rule

$$B_{ij} dx^1 \wedge \dots \wedge dx^7 = \frac{\partial}{\partial x^i} \lrcorner \varphi \wedge \frac{\partial}{\partial x^j} \lrcorner \varphi \wedge \varphi.$$

Then the metric g is defined by the formula

$$g_{ij} = \frac{1}{6^{\frac{2}{9}}} \frac{B_{ij}}{\det(B)^{\frac{1}{9}}}.$$

If one choose local coordinates such that in these coordinates $\varphi(p)$ will be expressed as associative form ϕ the metric will have an Euclidean form $g_{ij}(p) = \delta_{ij}$.

Definition 3 *If for the G_2 -structure $\nabla \varphi = 0$, where ∇ is a Levi-Civita connection of the metric $g = g_\varphi$, then (M, φ) is called a G_2 -manifold.*

Remark: *The equation $\nabla \varphi = 0$ is a very non-linear and thus it's so difficult to find on M a parallel G_2 -structure.*

Any G_2 -manifold has its holonomy group contained in G_2 . It's interesting that G_2 -manifolds are always Ricci-flat:

Theorem 1 (Bonan, [1])

$$\nabla\varphi = 0 \Rightarrow Ric(g_\varphi) = 0$$

There is a useful criteria for G_2 -structure to be parallel:

Theorem 2 (Fernandez and Gray, [2])

$$\nabla\varphi = 0 \Leftrightarrow d\varphi = 0 \text{ and } \delta\varphi = 0$$

where δ is the conjugate operator to the De Rham differential d w.r.t. the metric g_φ .

Definition 4 Let $\varphi = \varphi(t)$ be a smooth family of G_2 -structures on M . The flow of G_2 -structures is a system of evolutionary differential equations for components of φ in a basis $dx^i \wedge dx^j \wedge dx^k$:

$$\frac{\partial\varphi_{ijk}}{\partial t} = F(\varphi)_{ijk}$$

where $F(\varphi)_{ijk}$ — some, generally speaking, differential (in sense of only space variables) expressions on components of φ .

Examples of flows of G_2 -structures are

— the Laplacian flow [1] that was introduced by R. Bryant

$$\frac{\partial\varphi}{\partial t} = \Delta_g\varphi$$

where $\Delta_g = d\delta + \delta d$ is a Hodge Laplacian of the metric g_φ ,

— the General flow [3] introduced by S. Karigiannis

$$\frac{\partial\varphi_{ijk}}{\partial t} = h_i^l\varphi_{ljk} + h_j^l\varphi_{ilk} + h_k^l\varphi_{ijl} + X^l(*\varphi)_{lijk}$$

where h_{ij} is a symmetric tensor on M , X^k is a vector field on M , $*$ is a Hodge star operator of the metric g_φ .

Definition 5 The geometric flow of G_2 -structures on M is the equation

$$\frac{\partial\varphi}{\partial t} \wedge X = d\varphi$$

where X is a differential 1-form on M that does not depend on time.

3 Geometric flow on $C(S^3 \times S^3)$

Let's consider a cone over $S^3 \times S^3$. There are a frame of left-invariant vector fields on the Lie group $S^3 = SU(2)$ such that at the unit of $SU(2)$ it looks as follows

$$\xi^1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \xi^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \xi^3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

The Lie algebra generated by these vector fields has the multiplication law

$$[\xi^1, \xi^2] = 2\xi^3, \quad [\xi^2, \xi^3] = 2\xi^1, \quad [\xi^3, \xi^1] = 2\xi^2.$$

If η_1, η_2, η_3 are dual to ξ^1, ξ^2, ξ^3 1-forms then, by Cartan formula,

$$d\eta_1 = -2\eta_2 \wedge \eta_3, \quad d\eta_2 = -2\eta_3 \wedge \eta_1, \quad d\eta_3 = -2\eta_1 \wedge \eta_2.$$

Let $\eta_1, \eta_2, \eta_3, \tilde{\eta}_1$ and $\tilde{\eta}_2, \tilde{\eta}_3$ be left-invariant coframes on the first and on the second sphere of the Cartesian product $S^3 \times S^3$ correspondingly, let dr be a standart 1-form on \mathbb{R} . Now define the next 1-forms on $C(S^3 \times S^3)$

$$e^1 = A(r)(\eta_1 + \tilde{\eta}_1),$$

$$e^2 = A(r)(\eta_2 + \tilde{\eta}_2),$$

$$e^3 = A(r)(\eta_3 + \tilde{\eta}_3),$$

$$e^4 = B(r)(\eta_4 - \tilde{\eta}_4),$$

$$e^5 = B(r)(\eta_5 - \tilde{\eta}_5),$$

$$e^6 = B(r)(\eta_6 - \tilde{\eta}_6),$$

$$e^7 = dr$$

where $A(r)$ and $B(r)$ — some strictly positive functions, $r > 1$.

In the basis e^i , $i = 1, \dots, 7$ the G_2 -structure and the corresponding metric are given by the formulas

$$\varphi = e^{456} + e^{621} + e^{174} + e^{527} + e^{637} + e^{135} + e^{432}, \quad (1)$$

$$g = dr^2 + \sum_{i=1}^3 A^2(\eta_i + \tilde{\eta}_i)^2 + \sum_{j=1}^3 B^2(\eta_j - \tilde{\eta}_j)^2 \quad (2)$$

Parallelness of just described G_2 -structure was studied in [5].

Remark: *There is a one-to-one correspondence between the class of G_2 -structures 1 on $C(S^3 \times S^3)$ and the class of metrics 2 on $C(S^3 \times S^3)$.*

If A and B are functions smoothly depending on t , namely we deal with a smooth family of G_2 -structures

$$\begin{aligned}\varphi = \varphi(t) = & B^3(\eta_4 - \tilde{\eta}_4) \wedge (\eta_5 - \tilde{\eta}_5) \wedge (\eta_6 - \tilde{\eta}_6) + A^2 B(\eta_6 - \tilde{\eta}_6) \wedge (\eta_2 + \tilde{\eta}_2) \wedge (\eta_1 + \tilde{\eta}_1) \\ & + AB(\eta_1 + \tilde{\eta}_1) \wedge dr \wedge (\eta_4 - \tilde{\eta}_4) + AB(\eta_5 - \tilde{\eta}_5) \wedge (\eta_2 + \tilde{\eta}_2) \wedge dr + AB(\eta_6 - \tilde{\eta}_6) \wedge (\eta_3 + \tilde{\eta}_3) \wedge dr \\ & + A^2 B(\eta_1 + \tilde{\eta}_1) \wedge (\eta_3 + \tilde{\eta}_3) \wedge (\eta_5 - \tilde{\eta}_5) + A^2 B(\eta_4 - \tilde{\eta}_4) \wedge (\eta_3 + \tilde{\eta}_3) \wedge (\eta_2 + \tilde{\eta}_2),\end{aligned}$$

then we have a

Lemma 1 *The geometric flow*

$$\frac{\partial \varphi}{\partial t} \wedge dr = d\varphi \quad (3)$$

is equivalent to the system of PDE

$$\begin{cases} A = 2BB_x \\ 8BB_{xx} + 12B_x^2 = 1 \end{cases} \quad (4)$$

where $x = t + r$, $y = r - t$.

Proof of 1 is in Appendix.

Lemma 2 *General solution of 4 is given by the next expressions*

$$\begin{cases} A = 2B\sqrt{\frac{1}{12} + \frac{f(y)}{B^3}} \\ x = \int_0^B \frac{db}{\sqrt{\frac{1}{12} + \frac{f(y)}{b^3}}} + h(y) \end{cases} \quad (5)$$

where $f(y)$ and $h(y)$ are arbitrary smooth functions.

Proof: B and B_x are strictly positive functions, because any degeneration means that 2 is not a riemannian metric. The equation

$$8BB_{xx} + 12B_x^2 = 1, \quad (6)$$

is equivalent to

$$(B_x^2 B^3)_x = \frac{1}{12}(B^3)_x.$$

By integrating the last equation we get

$$B_x^2 = \frac{1}{12} + \frac{f(y)}{B^3} \quad (7)$$

where $f(y)$ — arbitrary smooth function.

Let's fix y' and construct solution of 6 along the characteristic $y = y'$. $B_x > 0$ so we have

$$\frac{dx}{dB} = \frac{1}{\sqrt{\frac{1}{12} + \frac{f(y')}{B^3}}}. \quad (8)$$

and

$$x = \int_0^B \frac{db}{\sqrt{\frac{1}{12} + \frac{f(y')}{b^3}}} + h(y').$$

This expression is valid for any y because y' was an arbitrary value:

$$x = \int_0^B \frac{db}{\sqrt{\frac{1}{12} + \frac{f(y)}{b^3}}} + h(y), \quad (9)$$

Combining 7 with the first equation of 4 we end the proof. \square

Definition 6 Let $g = g(t, r)$ be a continuous family of metrics on $C(S^3 \times S^3)$. We say that g converges to a metric g_∞ and write $g \rightarrow g_\infty$ as $t \rightarrow +\infty$, if

$$\forall K > 1 \sup_{1 < r \leq K} |g(r, t) - g_\infty(r)| \rightarrow 0 \text{ as } t \rightarrow +\infty. \quad (10)$$

The main result of this paper is the next

Theorem 3 For bounded $f(y)$ and $h(y)$ such that $f(y) \geq 0$, $h(y) < y$ the metric g corresponding to solution 5 of the flow 3 satisfy the next condition: $\frac{g}{(t+1)^2}$ converges to a conical metric $ds^2 + s^2 \cdot g_{S^3 \times S^3}$, $g_{S^3 \times S^3}$ — metric on $S^3 \times S^3$ that does not depend on s . In other words: homothety class of the metric g converges to a homothety class of g_∞ in the sense 10.

Proof: As $r > 1$, $t \geq 0$ then at each time t $x > t + 1$. Further, instead of variables x and y we will sometimes use $s = \frac{x}{t+1}$ and t , where $s > 1$ — a space variable of the limit metric. Let's prove that

$$\sup_{s>1} \left| \frac{B(s, t)}{t+1} - \frac{s}{\sqrt{12}} \right| \rightarrow 0, \text{ as } t \rightarrow +\infty \quad (11)$$

Norming by $t + 1$ is exactly homothety of corresponding metric.

By constituting $t = 0$ into 9 we get

$$r = \int_0^{B|_{t=0}(r)} \frac{db}{\sqrt{\frac{1}{12} + \frac{f(r)}{b^3}}} + h(r),$$

$$h(y) = y - \int_0^{B|_{t=0}(y)} \frac{db}{\sqrt{\frac{1}{12} + \frac{f(y)}{b^3}}}.$$

Remark: The condition $h(y) < y$ corresponds to positiveness of $B|_{t=0}$

By the mean value theorem for any $B > 0$ there exists B' such that

$$x = \int_0^B \frac{db}{\sqrt{\frac{1}{12} + \frac{f(y)}{b^3}}} + h(y) = \frac{B}{\sqrt{\frac{1}{12} + \frac{f(y)}{B'^3}}} + h(y).$$

If $x \rightarrow +\infty$ then $B, B' \rightarrow +\infty$ because $f(y) \geq 0$, $h(y)$ are bounded and $B_x \rightarrow \frac{1}{\sqrt{12}}$ as $B \rightarrow +\infty$.

So, when $x \approx +\infty$ $B \approx \frac{x}{\sqrt{12}} - \frac{h(y)}{\sqrt{12}}$

Let's justify the convergence 11. Firstly, let's notice that when t is fixed we have

$$\frac{\partial}{\partial s} \left(\frac{B(s, t)}{t+1} - \frac{s}{\sqrt{12}} \right) = \sqrt{\frac{1}{12} + \frac{f(y)}{B^3}} - \frac{1}{\sqrt{12}} \geq 0$$

$$\sqrt{\frac{1}{12} + \frac{f(y)}{B^3}} - \frac{1}{\sqrt{12}} \rightarrow 0, \text{ as } s = \frac{x}{t+1} \rightarrow +\infty \Leftrightarrow x \rightarrow +\infty$$

because g is bounded and ≥ 0 .

Thus we get

$$\sup_{s>1} \left| \frac{B(s, t)}{t+1} - \frac{s}{\sqrt{12}} \right| = \frac{|h(y)|}{\sqrt{12}(t+1)} \rightarrow 0 \text{ as } t \rightarrow +\infty \quad (12)$$

Now we can show that

$$\forall K > 1 \quad \sup_{1 < s \leq K} \left| \frac{B(s, t)^2}{(t+1)^2} - \frac{s^2}{12} \right| \rightarrow 0, \text{ as } t \rightarrow +\infty \quad (13)$$

$$\begin{aligned} \sup_{1 < s \leq K} \left| \frac{B(s, t)^2}{(t+1)^2} - \frac{s^2}{12} \right| &\leq \sup_{1 < s \leq K} \left| \frac{B(s, t)}{t+1} - \frac{s}{\sqrt{12}} \right| \left| \frac{B(s, t)}{t+1} + \frac{s}{\sqrt{12}} \right| \leq \\ &\frac{|h(y)|}{\sqrt{12}(t+1)} \left(\frac{|h(y)|}{\sqrt{12}(t+1)} + 2 \sup_{1 < s \leq K} \frac{s}{\sqrt{12}} \right) \rightarrow 0 \text{ as } t \rightarrow +\infty \end{aligned}$$

Finally, we prove the convergence 10 of the metric

$$\begin{aligned} \frac{g}{(t+1)^2} = \frac{1}{(t+1)^2} \left(dx^2 + \sum_{i=1}^3 A^2(\eta_i + \tilde{\eta}_i)^2 + \sum_{j=1}^3 B^2(\eta_j - \tilde{\eta}_j)^2 \right) = ds^2 + \sum_{i=1}^3 \frac{A^2}{(t+1)^2} (\eta_i + \tilde{\eta}_i)^2 \\ + \sum_{j=1}^3 \frac{B^2}{(t+1)^2} (\eta_j - \tilde{\eta}_j)^2 \end{aligned}$$

to the conical metric

$$ds^2 + \frac{s^2}{36} \sum_{i=1}^3 (\eta_i + \tilde{\eta}_i)^2 + \frac{s^2}{12} \sum_{j=1}^3 (\eta_j - \tilde{\eta}_j)^2.$$

Because the metrics are expressed in the same basis to finish the proof it's sufficient to keep in mind 13 and remember that $A = 2BB_x$. \square

Remark: *Conditions of the theorem 3 could be rewritten as some conditions on initial G_2 -structure (or metric) for the flow*

$$\frac{\partial \varphi}{\partial t} \wedge dr = d\varphi,$$

so we can establish Cauchy problem for the system

$$\begin{cases} \dot{B} + B' = \frac{A}{B} \\ \dot{A} + A' = \frac{1}{2}(1 - \frac{A^2}{B^2}) \end{cases} \quad (14)$$

Appendix

Let's compute the De Rham differential d of G_2 -structure φ on $C(S^3 \times S^3)$. To simplify notations we will denote by B' and A' partial derivatives $\frac{\partial B}{\partial r}$ and $\frac{\partial A}{\partial r}$ correspondingly. Recall that in coordinates described in the section 3

$$\varphi = e^{456} + e^{621} + e^{174} + e^{527} + e^{637} + e^{135} + e^{432},$$

$$d\varphi = de^{456} + de^{621} + de^{174} + de^{527} + de^{637} + de^{135} + de^{432}.$$

$$\begin{aligned} de^{456} &= de^4 \wedge e^5 \wedge e^6 - e^4 \wedge de^5 \wedge e^6 + e^4 \wedge e^5 \wedge de^6 = \left(\frac{B'}{B}e^7 \wedge e^4 + 2B(-\eta_2 \wedge \eta_3 + \tilde{\eta}_2 \wedge \tilde{\eta}_3)\right) \wedge e^5 \wedge e^6 \\ &- e^4 \wedge \left(\frac{B'}{B}e^7 \wedge e^5 + 2B(-\eta_3 \wedge \eta_1 + \tilde{\eta}_3 \wedge \tilde{\eta}_1)\right) \wedge e^6 + e^4 \wedge e^5 \wedge \left(\frac{B'}{B}e^7 \wedge e^6 + 2B(-\eta_1 \wedge \eta_2 + \tilde{\eta}_1 \wedge \tilde{\eta}_2)\right) = \\ &3\frac{B'}{B}e^7 \wedge e^4 \wedge e^5 \wedge e^6 + \frac{B}{2}\left(-\left(\frac{e^2}{A} + \frac{e^5}{B}\right) \wedge \left(\frac{e^3}{A} + \frac{e^6}{B}\right) + \left(\frac{e^2}{A} - \frac{e^5}{B}\right) \wedge \left(\frac{e^3}{A} - \frac{e^6}{B}\right)\right) \wedge e^5 \wedge e^6 - \frac{B}{2}e^4 \wedge \left(-\left(\frac{e^3}{A} + \frac{e^6}{B}\right) \wedge \left(\frac{e^1}{A} + \frac{e^4}{B}\right) + \right. \\ &\left. \left(\frac{e^3}{A} - \frac{e^6}{B}\right) \wedge \left(\frac{e^1}{A} - \frac{e^4}{B}\right)\right) \wedge e^6 + \frac{B}{2}e^4 \wedge e^5 \wedge \left(-\left(\frac{e^1}{A} + \frac{e^4}{B}\right) \wedge \left(\frac{e^2}{A} + \frac{e^5}{B}\right) + \left(\frac{e^1}{A} - \frac{e^4}{B}\right) \wedge \left(\frac{e^2}{A} - \frac{e^5}{B}\right)\right) = -3\frac{B'}{B}e^4 \wedge e^5 \wedge e^6 \wedge e^7 \end{aligned}$$

$$\begin{aligned} de^{621} &= de^6 \wedge e^2 \wedge e^1 - e^6 \wedge de^2 \wedge e^1 + e^6 \wedge e^2 \wedge de^1 = \left(\frac{B'}{B}e^7 \wedge e^6 + 2B(-\eta_1 \wedge \eta_2 + \tilde{\eta}_1 \wedge \tilde{\eta}_2)\right) \wedge e^2 \wedge e^1 \\ &- e^6 \wedge \left(\frac{A'}{A}e^7 \wedge e^2 + 2A(-\eta_3 \wedge \eta_1 - \tilde{\eta}_3 \wedge \tilde{\eta}_1)\right) \wedge e^1 + e^6 \wedge e^2 \wedge \left(\frac{A'}{A}e^7 \wedge e^1 + 2A(-\eta_2 \wedge \eta_3 - \tilde{\eta}_2 \wedge \tilde{\eta}_3)\right) = \\ &\frac{B'}{B}e^7 \wedge e^6 \wedge e^2 \wedge e^1 + 2\frac{A'}{A}e^7 \wedge e^6 \wedge e^2 \wedge e^1 + \frac{B}{2}\left(-\left(\frac{e^1}{A} + \frac{e^4}{B}\right) \wedge \left(\frac{e^2}{A} + \frac{e^5}{B}\right) + \left(\frac{e^1}{A} - \frac{e^4}{B}\right) \wedge \left(\frac{e^2}{A} - \frac{e^5}{B}\right)\right) \wedge e^2 \wedge e^1 - \frac{A}{2}e^6 \wedge \\ &\left(-\left(\frac{e^3}{A} + \frac{e^6}{B}\right) \wedge \left(\frac{e^1}{A} + \frac{e^4}{B}\right) - \left(\frac{e^3}{A} - \frac{e^6}{B}\right) \wedge \left(\frac{e^1}{A} - \frac{e^4}{B}\right)\right) \wedge e^1 + \frac{A}{2}e^6 \wedge e^2 \wedge \left(-\left(\frac{e^2}{A} + \frac{e^5}{B}\right) \wedge \left(\frac{e^3}{A} + \frac{e^6}{B}\right) - \right. \\ &\left. \left(\frac{e^2}{A} - \frac{e^5}{B}\right) \wedge \left(\frac{e^3}{A} - \frac{e^6}{B}\right)\right) = \frac{B'}{B}e^1 \wedge e^2 \wedge e^6 \wedge e^7 + 2\frac{A'}{A}e^1 \wedge e^2 \wedge e^6 \wedge e^7 \end{aligned}$$

$$\begin{aligned} de^{174} &= de^1 \wedge e^7 \wedge e^4 - e^1 \wedge de^7 \wedge e^4 + e^1 \wedge e^7 \wedge de^4 = \left(\frac{A'}{A}e^7 \wedge e^1 + 2A(-\eta_2 \wedge \eta_3 - \tilde{\eta}_2 \wedge \tilde{\eta}_3)\right) \wedge e^7 \wedge e^4 + e^1 \wedge e^7 \wedge \left(\frac{B'}{B}e^7 \wedge e^4 + 2B(-\eta_2 \wedge \eta_3 + \tilde{\eta}_2 \wedge \tilde{\eta}_3)\right) = \\ &\frac{A}{2}\left(-\left(\frac{e^2}{A} + \frac{e^5}{B}\right) \wedge \left(\frac{e^3}{A} + \frac{e^6}{B}\right) - \left(\frac{e^2}{A} - \frac{e^5}{B}\right) \wedge \left(\frac{e^3}{A} - \frac{e^6}{B}\right)\right) \wedge e^7 \wedge e^4 + \frac{B}{2}e^1 \wedge e^7 \wedge \left(-\left(\frac{e^2}{A} + \frac{e^5}{B}\right) \wedge \left(\frac{e^3}{A} + \frac{e^6}{B}\right) + \right. \\ &\left. \left(\frac{e^2}{A} - \frac{e^5}{B}\right) \wedge \left(\frac{e^3}{A} - \frac{e^6}{B}\right)\right) = -\frac{1}{A}e^2 \wedge e^3 \wedge e^7 \wedge e^4 - \frac{A}{B^2}e^5 \wedge e^6 \wedge e^7 \wedge e^4 - \frac{1}{A}e^1 \wedge e^7 \wedge e^2 \wedge e^6 - \frac{1}{A}e^1 \wedge e^7 \wedge e^5 \wedge e^3 = \\ &\frac{1}{A}e^2 \wedge e^3 \wedge e^4 \wedge e^7 + \frac{A}{B^2}e^4 \wedge e^5 \wedge e^6 \wedge e^7 - \frac{1}{A}e^1 \wedge e^2 \wedge e^6 \wedge e^7 + \frac{1}{A}e^1 \wedge e^3 \wedge e^5 \wedge e^7 \end{aligned}$$

$$\begin{aligned} de^{527} &= de^5 \wedge e^2 \wedge e^7 - e^5 \wedge de^2 \wedge e^7 + e^5 \wedge e^2 \wedge de^7 = \left(\frac{B'}{B}e^7 \wedge e^5 + 2B(-\eta_3 \wedge \eta_1 + \tilde{\eta}_3 \wedge \tilde{\eta}_1)\right) \wedge e^2 \wedge e^7 - e^5 \wedge \left(\frac{A'}{A}e^7 \wedge e^2 + 2A(-\eta_3 \wedge \eta_1 - \tilde{\eta}_3 \wedge \tilde{\eta}_1)\right) \wedge e^7 = \\ &\frac{B}{2}\left(-\left(\frac{e^3}{A} + \frac{e^6}{B}\right) \wedge \left(\frac{e^1}{A} + \frac{e^4}{B}\right) + \left(\frac{e^3}{A} - \frac{e^6}{B}\right) \wedge \left(\frac{e^1}{A} - \frac{e^4}{B}\right)\right) \wedge e^2 \wedge e^7 + \frac{A}{2}e^5 \wedge \left(\left(\frac{e^3}{A} + \frac{e^6}{B}\right) \wedge \left(\frac{e^1}{A} + \frac{e^4}{B}\right) + \right. \\ &\left. \left(\frac{e^3}{A} - \frac{e^6}{B}\right) \wedge \left(\frac{e^1}{A} - \frac{e^4}{B}\right)\right) \wedge e^7 = -\frac{1}{A}e^3 \wedge e^4 \wedge e^2 \wedge e^7 - \frac{1}{A}e^6 \wedge e^1 \wedge e^2 \wedge e^7 + \frac{1}{A}e^5 \wedge e^3 \wedge e^1 \wedge e^7 + \frac{A}{B^2}e^5 \wedge e^6 \wedge e^4 \wedge e^7 = \\ &-\frac{1}{A}e^2 \wedge e^3 \wedge e^4 \wedge e^7 - \frac{1}{A}e^1 \wedge e^2 \wedge e^6 \wedge e^7 - \frac{1}{A}e^1 \wedge e^3 \wedge e^5 \wedge e^7 + \frac{A}{B^2}e^4 \wedge e^5 \wedge e^6 \wedge e^7 \end{aligned}$$

$$\begin{aligned} de^{637} &= de^6 \wedge e^3 \wedge e^7 - e^6 \wedge de^3 \wedge e^7 + e^6 \wedge e^3 \wedge de^7 = \left(\frac{B'}{B}e^7 \wedge e^6 + 2B(-\eta_1 \wedge \eta_2 + \tilde{\eta}_1 \wedge \tilde{\eta}_2)\right) \wedge e^3 \wedge e^7 - e^6 \wedge \left(\frac{A'}{A}e^7 \wedge e^3 + 2A(-\eta_1 \wedge \eta_2 - \tilde{\eta}_1 \wedge \tilde{\eta}_2)\right) \wedge e^7 = \\ &\frac{B}{2}\left(-\left(\frac{e^1}{A} + \frac{e^4}{B}\right) \wedge \left(\frac{e^2}{A} + \frac{e^5}{B}\right) + \left(\frac{e^1}{A} - \frac{e^4}{B}\right) \wedge \left(\frac{e^2}{A} - \frac{e^5}{B}\right)\right) \wedge e^3 \wedge e^7 + \frac{A}{2}e^6 \wedge \left(\left(\frac{e^1}{A} + \frac{e^4}{B}\right) \wedge \left(\frac{e^2}{A} + \frac{e^5}{B}\right) + \right. \\ &\left. \left(\frac{e^1}{A} - \frac{e^4}{B}\right) \wedge \left(\frac{e^2}{A} - \frac{e^5}{B}\right)\right) \wedge e^7 = -\frac{1}{A}e^1 \wedge e^5 \wedge e^3 \wedge e^7 - \frac{1}{A}e^4 \wedge e^2 \wedge e^3 \wedge e^7 + \frac{1}{A}e^6 \wedge e^1 \wedge e^2 \wedge e^7 + \frac{A}{B^2}e^6 \wedge e^4 \wedge e^5 \wedge e^7 = \\ &\frac{1}{A}e^1 \wedge e^3 \wedge e^5 \wedge e^7 - \frac{1}{A}e^2 \wedge e^3 \wedge e^4 \wedge e^7 + \frac{1}{A}e^1 \wedge e^2 \wedge e^6 \wedge e^7 + \frac{A}{B^2}e^4 \wedge e^5 \wedge e^6 \wedge e^7 \end{aligned}$$

$$\begin{aligned} de^{135} &= de^1 \wedge e^3 \wedge e^5 - e^1 \wedge de^3 \wedge e^5 + e^1 \wedge e^3 \wedge de^5 = \left(\frac{A'}{A}e^7 \wedge e^1 + 2A(-\eta_2 \wedge \eta_3 - \tilde{\eta}_2 \wedge \tilde{\eta}_3)\right) \wedge e^3 \wedge e^5 - e^1 \wedge \left(\frac{A'}{A}e^7 \wedge e^3 + 2A(-\eta_1 \wedge \eta_2 - \tilde{\eta}_1 \wedge \tilde{\eta}_2)\right) \wedge e^5 + e^1 \wedge e^3 \wedge \left(\frac{B'}{B}e^7 \wedge e^5 + 2B(-\eta_3 \wedge \eta_1 + \tilde{\eta}_3 \wedge \tilde{\eta}_1)\right) = \end{aligned}$$

$$\begin{aligned} & \frac{B'}{B}e^7 \wedge e^1 \wedge e^3 \wedge e^5 + 2\frac{A'}{A}e^7 \wedge e^1 \wedge e^3 \wedge e^5 + \frac{A}{2}(-(\frac{e^2}{A} + \frac{e^5}{B}) \wedge (\frac{e^3}{A} + \frac{e^6}{B}) - (\frac{e^2}{A} - \frac{e^5}{B}) \wedge (\frac{e^3}{A} - \frac{e^6}{B})) \wedge e^3 \wedge e^5 + \frac{A}{2}e^1 \wedge \\ & ((\frac{e^1}{A} + \frac{e^4}{B}) \wedge (\frac{e^2}{A} + \frac{e^5}{B}) + (\frac{e^1}{A} - \frac{e^4}{B}) \wedge (\frac{e^2}{A} - \frac{e^5}{B})) \wedge e^5 + \frac{B}{2}e^1 \wedge e^3 \wedge (- (\frac{e^3}{A} + \frac{e^6}{B}) \wedge (\frac{e^1}{A} + \frac{e^4}{B}) + (\frac{e^3}{A} - \frac{e^6}{B}) \wedge (\frac{e^1}{A} - \frac{e^4}{B})) = \\ & -\frac{B'}{B}e^1 \wedge e^3 \wedge e^5 \wedge e^7 - 2\frac{A'}{A}e^1 \wedge e^3 \wedge e^5 \wedge e^7 \end{aligned}$$

$$\begin{aligned} de^{432} &= de^4 \wedge e^3 \wedge e^2 - e^4 \wedge de^3 \wedge e^2 + e^4 \wedge e^3 \wedge de^2 = (\frac{B'}{B}e^7 \wedge e^4 + 2B(-\eta_2 \wedge \eta_3 + \tilde{\eta}_2 \wedge \tilde{\eta}_3)) \wedge e^3 \wedge \\ & e^2 - e^4 \wedge (\frac{A'}{A}e^7 \wedge e^3 + 2A(-\eta_1 \wedge \eta_2 - \tilde{\eta}_1 \wedge \tilde{\eta}_2)) \wedge e^2 + e^4 \wedge e^3 \wedge (\frac{A'}{A}e^7 \wedge e^2 + 2A(-\eta_3 \wedge \eta_1 - \tilde{\eta}_3 \wedge \tilde{\eta}_1)) = \\ & \frac{B'}{B}e^7 \wedge e^4 \wedge e^3 \wedge e^2 + 2\frac{A'}{A}e^7 \wedge e^4 \wedge e^3 \wedge e^2 + \frac{B}{2}(-(\frac{e^2}{A} + \frac{e^5}{B}) \wedge (\frac{e^3}{A} + \frac{e^6}{B}) + (\frac{e^2}{A} - \frac{e^5}{B}) \wedge (\frac{e^3}{A} - \frac{e^6}{B})) \wedge e^3 \wedge e^2 + \frac{A}{2}e^4 \wedge \\ & ((\frac{e^1}{A} + \frac{e^4}{B}) \wedge (\frac{e^2}{A} + \frac{e^5}{B}) + (\frac{e^1}{A} - \frac{e^4}{B}) \wedge (\frac{e^2}{A} - \frac{e^5}{B})) \wedge e^2 - \frac{A}{2}e^4 \wedge e^3 \wedge ((\frac{e^3}{A} + \frac{e^6}{B}) \wedge (\frac{e^1}{A} + \frac{e^4}{B}) + (\frac{e^3}{A} - \frac{e^6}{B}) \wedge (\frac{e^1}{A} - \frac{e^4}{B})) = \\ & \frac{B'}{B}e^2 \wedge e^3 \wedge e^4 \wedge e^7 + 2\frac{A'}{A}e^2 \wedge e^3 \wedge e^4 \wedge e^7 \end{aligned}$$

$$\begin{aligned} d\varphi &= -3\frac{B'}{B}e^4 \wedge e^5 \wedge e^6 \wedge e^7 + \frac{B'}{B}e^1 \wedge e^2 \wedge e^6 \wedge e^7 + 2\frac{A'}{A}e^1 \wedge e^2 \wedge e^6 \wedge e^7 + \frac{1}{A}e^2 \wedge e^3 \wedge e^4 \wedge e^7 + \frac{A}{B^2}e^4 \wedge e^5 \wedge \\ & e^6 \wedge e^7 - \frac{1}{A}e^1 \wedge e^2 \wedge e^6 \wedge e^7 + \frac{1}{A}e^1 \wedge e^3 \wedge e^5 \wedge e^7 - \frac{1}{A}e^2 \wedge e^3 \wedge e^4 \wedge e^7 - \frac{1}{A}e^1 \wedge e^2 \wedge e^6 \wedge e^7 - \frac{1}{A}e^1 \wedge e^3 \wedge e^5 \wedge \\ & e^7 + \frac{A}{B^2}e^4 \wedge e^5 \wedge e^6 \wedge e^7 + \frac{1}{A}e^1 \wedge e^3 \wedge e^5 \wedge e^7 - \frac{1}{A}e^2 \wedge e^3 \wedge e^4 \wedge e^7 + \frac{1}{A}e^1 \wedge e^2 \wedge e^6 \wedge e^7 + \frac{A}{B^2}e^4 \wedge e^5 \wedge e^6 \wedge e^7 - \\ & \frac{B'}{B}e^1 \wedge e^3 \wedge e^5 \wedge e^7 - 2\frac{A'}{A}e^1 \wedge e^3 \wedge e^5 \wedge e^7 + \frac{B'}{B}e^2 \wedge e^3 \wedge e^4 \wedge e^7 + 2\frac{A'}{A}e^2 \wedge e^3 \wedge e^4 \wedge e^7 = 3(\frac{A}{B^2} - \frac{B'}{B})e^4 \wedge e^5 \wedge \\ & e^6 \wedge e^7 + (\frac{B'}{B} + 2\frac{A'}{A} - \frac{1}{A})e^1 \wedge e^2 \wedge e^6 \wedge e^7 + (\frac{B'}{B} + 2\frac{A'}{A} - \frac{1}{A})e^2 \wedge e^3 \wedge e^4 \wedge e^7 - (\frac{B'}{B} + 2\frac{A'}{A} - \frac{1}{A})e^1 \wedge e^3 \wedge e^5 \wedge e^7 \end{aligned}$$

Let's compute now $\frac{\partial \varphi}{\partial t}$. Recall that

$$e^i = A(\eta_i + \tilde{\eta}_i) \text{ for } i = 1, 2, 3$$

$$e^j = B(\eta_j - \tilde{\eta}_j) \text{ for } j = 4, 5, 6$$

$$e^7 = dr.$$

Then

$$\begin{aligned} \frac{\partial e^{456}}{\partial t} &= 3\frac{\dot{B}}{B}e^{456} \\ \frac{\partial e^{621}}{\partial t} &= (\frac{\dot{B}}{B} + 2\frac{\dot{A}}{A})e^{621} \\ \frac{\partial e^{174}}{\partial t} &= (\frac{\dot{B}}{B} + \frac{\dot{A}}{A})e^{174} \\ \frac{\partial e^{527}}{\partial t} &= (\frac{\dot{B}}{B} + \frac{\dot{A}}{A})e^{527} \\ \frac{\partial e^{637}}{\partial t} &= (\frac{\dot{B}}{B} + \frac{\dot{A}}{A})e^{637} \\ \frac{\partial e^{135}}{\partial t} &= (\frac{\dot{B}}{B} + 2\frac{\dot{A}}{A})e^{135} \\ \frac{\partial e^{432}}{\partial t} &= (\frac{\dot{B}}{B} + 2\frac{\dot{A}}{A})e^{432} \end{aligned}$$

where \dot{A} and \dot{B} are partial w.r.t. t derivatives of functions A and B correspondingly.

Proof of the lemma 1:

Keeping in mind above calculations we have

$$3\frac{\dot{B}}{B}e^{4567} - (\frac{\dot{B}}{B} + 2\frac{\dot{A}}{A})e^{1267} + (\frac{\dot{B}}{B} + 2\frac{\dot{A}}{A})e^{1357} - (\frac{\dot{B}}{B} + 2\frac{\dot{A}}{A})e^{2347} = \frac{\partial\varphi}{\partial t} \wedge e^7 = d\varphi = 3(\frac{A}{B^2} - \frac{B'}{B})e^{4567} + (\frac{B'}{B} + 2\frac{A'}{A} - \frac{1}{A})e^{1267} + (\frac{B'}{B} + 2\frac{A'}{A} - \frac{1}{A})e^{2347} - (\frac{B'}{B} + 2\frac{A'}{A} - \frac{1}{A})e^{1357}$$

or

$$\begin{cases} \frac{\dot{B}}{B} = -\frac{B'}{B} + \frac{A}{B^2} \\ 2\frac{\dot{A}}{A} + \frac{\dot{B}}{B} = \frac{1}{A} - 2\frac{A'}{A} - \frac{B'}{B} \end{cases}$$

or

$$\begin{cases} \dot{B} + B' = \frac{A}{B} \\ \dot{A} + A' = \frac{1}{2}(1 - \frac{A^2}{B^2}) \end{cases} \quad (15)$$

Let's change variables t and r on $x = r + t$ and $y = r - t$. In these variables equations 15 look as follow

$$\begin{cases} 2B_x = \frac{A}{B} \\ 2A_x = \frac{1}{2}(1 - \frac{A^2}{B^2}) \end{cases} \quad (16)$$

By expressing the function A from the first equation and substituting it into the second equation of 16 we get the desired system

$$\begin{cases} A = 2BB_x \\ 8BB_{xx} + 12B_x^2 = 1 \end{cases}$$

□

1. *Edmond Bonan* Sur les variétés Riemanniennes à groupe d'holonomie G_2 ou $\text{Spin}(7)$, C. R. Acad. Sci. Paris 262 (1966).
2. *Marisa Fernández; Alfred Gray* Riemannian manifolds with structure group G_2 , Ann. Mat. Pura Appl. 4 132 (1982), 19–45.
3. *Bryant, Robert L.* Some remarks on G_2 -structures, Proceeding of Gokova Geometry-Topology Conference 2005 edited by S. Akbulut, T. Onder, and R.J. Stern (2006), International Press, 75–109.
4. *Spiro Karigiannis* Flows of G_2 -Structures, I, Quarterly Journal of Mathematics, 60 (2009), 487–522.
5. *Ya.V. Bazaikin, O.A. Bogoyavlenskaya* Complete Riemannian G_2 Holonomy Metrics on Deformations of Cones over $S^3 \times S^3$, arXiv:1301.6379 [math.DG]

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